James Clerk Maxwell (1831-1879)

- Born in Edinburgh, Scotland.

- A humble, deeply religious man.

- A child prodigy in mathematics, he contributed an original paper to the Edinburgh Society at the age of 15.

- He developed the kinetic theory of gases in 1859-1860, showing that gas molecules have a distribution of speeds at any temperature.

- In 1873 he published his Treatise on Electricity and Magnetism - proposing the fundamental equations now called “Maxwell’s equations” relating electric and magnetic fields.

- He suggested that light is an electromagnetic wave, later confirmed by Hertz (1888).

- Maxwell’s relations in thermodynamics.
Maxwell’s Demon

• Invented in 1867, in a letter to the physicist Peter G. Tait. An imaginary being.

• **Theory of Heat, 1871:**

  “let us suppose that ... a vessel is divided into two portions, A and B, by a division in which there is a small hole, and that a being, who can see the individual molecules, opens and closes this hole, so as to allow only the swifter molecules to pass from A to B, and only the slower ones to pass from B to A. He will thus, without the expenditure of work, raise the temperature of B and lower that of A, in contradiction to the second law of thermodynamics.”

• Modern theories deny the possibility of such action, and depend on the information necessary to detect the molecules.
Stirling's Approximation

- The value of $N!$ increases enormously for large $N$. (Try $70!$ and $100!$ on your calculator.)

- In 1730 the Scottish mathematician James Stirling (1692-1770) proposed the following approximations for large $N$:

$$N! \approx N^N e^{-N} \sqrt{2\pi N}$$

$$\ln(N!) \approx N \ln N - N \quad \leftarrow \text{(the most useful form)}$$

- This becomes a very good approximation as $N$ becomes large:

<table>
<thead>
<tr>
<th>$N$</th>
<th>$N!$</th>
<th>Actual $\ln(N!)$</th>
<th>Stirling $\ln(N!)$</th>
<th>Error</th>
<th>% Rel. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$3.629 \times 10^6$</td>
<td>15.10</td>
<td>13.03</td>
<td>2.07</td>
<td>13.7%</td>
</tr>
<tr>
<td>100</td>
<td>$9.333 \times 10^{157}$</td>
<td>363.74</td>
<td>360.52</td>
<td>3.22</td>
<td>0.89%</td>
</tr>
<tr>
<td>1000</td>
<td>$4.024 \times 10^{2567}$</td>
<td>5,912.13</td>
<td>5,907.76</td>
<td>4.37</td>
<td>0.074%</td>
</tr>
<tr>
<td>5000</td>
<td>--</td>
<td>37,591.14</td>
<td>37,585.97</td>
<td>5.17</td>
<td>0.013%</td>
</tr>
</tbody>
</table>
The Method of Lagrange Multipliers

- The Lagrange method of undetermined multipliers was introduced by the French mathematician Joseph Louis Lagrange (1736-1813).

- The problem: To find extrema (maxima or minima) of a function $f(x_1,x_2,x_3,...,x_n)$ under constraints $g_1(x_1,x_2,...,x_n) = 0$, $g_2(x_1,x_2,...,x_n)= 0$, etc.

- Solution: One solves the set of $n$ equations

$$
\frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial g_1}{\partial x_1} + \lambda_2 \frac{\partial g_2}{\partial x_1} + \lambda_3 \frac{\partial g_3}{\partial x_1} + ... = 0
$$

$$
\frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial g_1}{\partial x_2} + \lambda_2 \frac{\partial g_2}{\partial x_2} + \lambda_3 \frac{\partial g_3}{\partial x_2} + ... = 0
$$

... etc., where the $\lambda_i$ are the “undetermined multipliers”.
Example of the use of Lagrange Multipliers

- Problem: To maximize the area $A = xy$ of a rectangle under the constraint that its perimeter $P = 2x + 2y$ is fixed. (Thus $f = xy$ and the constraint is $g = 2x + 2y - P_0 = 0$.)

- Solution: The equations are

$$\frac{\partial A}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \quad y + \lambda \cdot 2 = 0 \quad \lambda = -y/2$$

$$\frac{\partial A}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \quad x + (-y/2) \cdot 2 = 0 \quad x = y$$

- Conclusion: When a rectangle has a constant perimeter $P$, the maximum area is achieved if the two sides are equal (i.e., it's a square). A square has the maximum area in this case.
Second example of Lagrange Multipliers

- Suppose we want to maximize the area of a rectangle inscribed within a circle of radius 1.

- From the diagram, the area is \( A = 4xy \), and the constraint is \( g = x^2 + y^2 - 1 = 0 \).

- With \( f + \lambda g = 4xy + \lambda(x^2 + y^2 - 1) \), the equations are:

\[
\begin{align*}
4y + 2\lambda x &= 0 \\
4x + \left(-\frac{2y}{x}\right)2y &= 0
\end{align*}
\]

\[
\lambda = -\frac{2y}{x}
\]

- So \( \{4x - 4y^2/x\} = 0 \) and \( x^2 - y^2 = 0 \). Thus again the maximum area occurs for a square, \( x = y \). Also, since the constraint is \( x^2 + y^2 = 1, 2x^2 = 1 \), and \( x = y = 1/\sqrt{2} = 0.707 \).

- Note that the area of the rectangle is \( 4xy = 2 \) and the area of the circle is \( \pi r^2 = \pi \). So the ratio of the areas is \( 2/\pi = 0.637 \).